Nonlocal theory for fractional kinetic equations

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A bit of history (of Modena also)

Consider classical kinetic (or Fokker-Planck) equations

$$\underbrace{(\partial_t + v \cdot \nabla_x)}_{\text{transport}} f = \underbrace{\nabla_v \cdot (A \nabla_v f)}_{\text{diffusion}} + \underbrace{B \cdot \nabla_v f + h}_{\text{source}} \qquad \underbrace{(t, x, v)}_{\text{times,space,velocity}} \quad \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$$

under natural ellipticity conditions on the coefficients and scalar field

$$\begin{cases} 0 < \lambda \mathbb{I}_n \leq A \leq \Lambda \mathbb{I}_n \\ |B| \leq \Lambda \\ h \text{ essentially bounded} \end{cases}$$

In this scenarion, classical first Hölder regularity in the spirit of the works of De Giorgi-Nash & Moser (*DGNM* in short) theroy is completed

- Pascucci & Polidoro, *CCM* (2004): L^∞-L^2 estimates via Moser's Iteration.
- Golse, Imbert, Mouhot & Vasseur, *Ann. SNS* (2019): Hölder regularity + Harnack inequality
- Guerand & Mouhot, *JEP* (2021): Weak Harnack inequality.
- Anceschi & Rebucci, *JDE* (2022): Weak regularity for kinetic equations with more than one spatial commutator.
- Anceschi et. al, *Preprint* (2024): Poincaré inequality based on local trajectories and weak Harnack

What's happen is the diffusion is nonlocal??

We investigate local properties of solutions $f \equiv f(t,x,v)$ to a wide class of integro-differential equations having as toy model

$$(\partial_t + v \cdot \nabla_x) f + (-\triangle_v)^s f = 0, \qquad (t,x,v) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$$

Here we denote with $(-\triangle_v)^s$ the fractional Laplacian acting only the velocity variables

$$(-\triangle_v)^s f(t,x,v) = p.\,v. \int_{\mathbb{R}^n} \frac{f(t,x,v) - f(t,x,w)}{|v-w|^{n+2s}}\,\mathrm{d}w \qquad s \in (0,1).$$

What are we looking for?

We want to prove classical results in the spirit of the *DGNM* theory for elliptic and parabolic equations.

Why are we interested in nonlocal kinetic equations?

Consider the Boltzmann equation

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{Q}(f, f).$$

Under special circumstances the nonlinear and nonlocal collision operator $\mathcal{Q}(\cdot)$ can be "linearized" as follows

$$\mathcal{Q}(f, f) \approx \mathcal{L}_{K_f} f + \langle \text{lower order terms} \rangle$$

where

$$\mathcal{L}_{K_f} f(t, x, v) := p.\,v. \int_{\mathbb{R}^n} \big(f(t, x, w) - f(t, x, v) \big) K_f(t, x, v, w) \, \mathrm{d}w$$

with the measurable kernel depending on the solution f itself.

Question:

Is it possible to apply ideas from the area of integro-differential equations in the context of the Boltzmann equation.

There are several difficulties that we must overcome:

- In what way is the kernel K_f elliptic?
- Is it possible to extend the regularity results for integro-differential parabolic equations to the setting of the Boltzmann equations?

Solution class

Conditional regime

There are m_o, M_o, E_o, $H_\mathrm{o} > 0$ such that for all (t, x) it holds:

$$\begin{cases} m_\mathrm{o} \leq \int_{\mathbb{R}^n} f(t,x,v)\,\mathrm{d}v \leq M_\mathrm{o} \\ \int_{\mathbb{R}^n} f(t,x,v)|v|^2\,\mathrm{d}v \leq E_\mathrm{o} \\ \int_{\mathbb{R}^n} f(t,x,v)\log f(t,x,v)\,\mathrm{d}v \leq H_\mathrm{o} \end{cases} \tag{H}$$

Under the bounds (H) the operator $\mathcal{L}_{K_f}$ satisfies:

$$\text{(H)} \quad \Longrightarrow \quad \begin{cases} \text{Coercivity: } \|f\|^2_{\mathring{H}^s} \lesssim -\int_{\mathbb{R}^n} \mathcal{L}_{K_f} f(v) f(v)\,\mathrm{d}v + \|f\|^2_{L^2} \\ \text{Non-degeneracy assumption when } 0 < s < 1/2 \\ \text{Weak upper bounds } \int_{\mathbb{R}^n \setminus B_r(v)} K_f(v,w')\,\mathrm{d}w \lesssim r^{-2s} \\ \text{Cancellation conditions} \end{cases}$$

Remark

The ellipticity constants of K_f depends on the bounds on the quantities in (H) *and on $\|f\|_{L^\infty}$.*

De Giorgi methods for the Boltzmann equation

Theorem (Imbert & Silvestre, *JEMS* (2020))

There exists universal constants $R_o \in (0,1)$, $R_1 > 1$ *and* $\zeta \in (0,1)$ *such that if* f *is a nonnegative weak supersolutions to*

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{L}_{K_f} f + h \quad \text{in } (-1,0] \times B_{R_1^{1+2s}} \times B_{R_1}$$

then

$$\|f\|_{L^\zeta(Q^-)} \lesssim \inf_{Q^+} f + \|h\|_{L^\infty},$$

with

$$Q^+ := (-R_o^{2s}, 0] \times B_{R_o^{1+2s}} \times B_r$$
$$\text{and} \quad Q^- := (-1, -1 + R_o{}^{2s}] \times B_{R_o{}^{1+2s}} \times B_{R_o}$$

- The weak Harnack inequality is enough to dervie Hölder regularity for solutions:

$$\|f\|_{C^\alpha((-1/2,0)\times B_{1/2} \times B_{1/2})} \lesssim \|f\|_{L^\infty((-1,0]\times B_1 \times \mathbb{R}^n)} + \|h\|_{L^\infty((-1,0]\times B_1 \times B_1)}$$

- Both the full Harnack inequality and a linear L^∞-L^2 do not hold for nonlocal equations as the one modeling the Boltzmann equation.

Nonlocal Fokker-Planck equations

Consider a “better” equation

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{L}_K f + h \tag{1}$$

with the measurable kernel $K : \mathbb{R}^{1+3n} \to [0, +\infty]$ satisfying

$$\begin{cases} K(t,x,v,w) = K(t,x,w,v), \\ K(t,x,v,w) \approx |v-w|^{-n-2s}. \end{cases}$$

- Stokols, *SIMA* (2019): Hölder regularity for essentially bounded solutions
- Loher, *JFA* (2024): “Not-so-Strong” Harnack for essentially bounded solutions

$$\sup_{Q^-} f \lesssim \left(\inf_{Q^+} f \right)^\beta \qquad \beta \in (0,1)$$

- Anceschi et al. *arXiv* (2024): weak Harnack inequality via Poincaré inequality.

Remark

The classical full Harnack inequality is absent in all of these previous works

Motivation:

Develop the nonlocal theory of Kolmogorov equations completing the key missing results (full Harnack inequality and L^∞-L^2 estimate).

Remark

The transport and diffusion defines a geometric structure which leaves them invariant and preserves their homogeneity.

Endow $\mathbb{R}^{1+2n}$ with the following Galilean transformation and scalings

$$\begin{cases} z_o \circ z := (t + t_o,\, x + x_o + t v_o,\, v + v_o) \\ D_R(z) := (R^{2s} t,\, R^{1+2s} x,\, Rv) \quad \forall R > 0 \end{cases}$$

We introduce a family of domains respecting the invariant transformations defined above. Given

$$Q_1 \equiv Q_1(0) := U_1(0,0) \times B_1(0) = (-1,0] \times B_1(0) \times B_1(0)\,.$$

define the *slanted* cylinder $Q_R(z_o)$ by

$$\begin{aligned} Q_R(z_o) \;&:=\; \{z_o \circ D_R(z) \,:\, z \in Q_1\} \\ &\equiv\; \Big\{(t,x,v) :\; t_o - R^{2s} < t \leq t_o, \\ &\qquad |x - x_o - (t - t_o) v_o| < R^{1+2s},\; |v - v_o| < R\Big\}. \end{aligned}$$

Functional setting

For $s \in (0,1)$ we denote with $W^{s,2}(\mathcal{A})$ the classical fractional Sobolev space

$$W^{s,2}(\mathcal{A}) := \left\{ f \in L^2(\mathcal{A}) \ : \ [f]_{s,2;\mathcal{A}} < +\infty \right\} ,$$

where

$$[f]_{s,2;\mathcal{A}} := \left(\int_{\mathcal{A}} \int_{\mathcal{A}} \frac{|f(v) - f(w)|^2}{|v - w|^{n+2s}} \, \mathrm{d}v \, \mathrm{d}w \right)^{1/2} ,$$

equipped with the usual norm

$$\|f\|_{W^{s,2}(\mathcal{A})} := \|f\|_{L^2(\mathcal{A})} + [f]_{s,2;\mathcal{A}} \, .$$

Definition (Di Castro, Kuusi & Palatucci, *JFA* (2014) + *Ann. IHP-C* (2016))

Let f be a measurable function on $(t_1, t_2) \times \Omega_x \times \mathbb{R}^n \subset \mathbb{R}^{1+2n}$, then the nonlocal tail of f centered in $v_\circ$ and of radius r is defined by

$$\mathrm{Tail}(f; v_\circ, R) \ := \ R^{2s} \int_{\mathbb{R}^n \setminus B_R(v_\circ)} |f(v)| |v_\circ - v|^{-n-2s} \, \mathrm{d}v.$$

In connections with the nonlocal tail consider the related tail space

$$L^1_{2s}(\mathbb{R}^n) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \ : \ \int_{\mathbb{R}^n} \frac{|f(v)|}{(1 + |v|)^{n+2s}} \, \mathrm{d}v < \infty \right\}$$

Weak formulation

Given $\Omega := (t_1, t_2) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$ denote by

$$\mathcal{W} := \Big\{ f \in L^2_{\mathrm{loc}}((t_1, t_2) \times \Omega_x;\, W^{s,2}_{\mathrm{loc}}(\Omega_v)) \cap L^1_{\mathrm{loc}}((t_1, t_2) \times \Omega_x;\, L^1_{2s}(\mathbb{R}^n)) \\ : (\partial_t + v \cdot \nabla_x) f \in L^2_{\mathrm{loc}}((t_1, t_2) \times \Omega_x;\, W^{-s,2}(\mathbb{R}^n)) \Big\}.$$

and by $\mathcal{E}^K(\cdot)$ the nonlocal energy

$$\mathcal{E}^K(f, \phi) := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \big(f(v) - f(w)\big)\big(\phi(v) - \phi(w)\big) K(v, w)\, \mathrm{d}v\, \mathrm{d}w.$$

Definition

A function $f \in \mathcal{W}$ is a weak subsolution (resp., supersolution) to (1) in Ω if

$$\int_{t_1}^{t_2} \int_{\Omega_x} \mathcal{E}^K(f, \phi)\, \mathrm{d}x\, \mathrm{d}t + \int_{\Omega} (\partial_t f + v \cdot \nabla_x f) \phi \, \mathrm{d}z \overset{(\geq \text{ resp.})}{\leq} \int_{\Omega} h \, \phi \, \mathrm{d}z$$

for any nonnegative $\phi \in L^2_{loc}((t_1, t_2) \times \Omega_x; W^{s,2}(\mathbb{R}^n))$ such that $\phi(t, x, \cdot)$ is compactly supported in Ω_v. A function $f \in \mathcal{W}$ is a weak solution if it is both a weak sub- and supersolution.

A roadmap to get the full Harnack

Consider

$$(-\triangle_v)^s f = 0 \qquad \text{in } B_{2R}(0) \subset \mathbb{R}^n. \tag{2}$$

To get the full Harnack inequality for solutions $f \geq 0$ to (2) it sufficies to combine that subsolutions satisfies

$$\sup_{B_{R/2}(0)} f_+ \leq c(\delta)\|f_+\|_{L^2(B_R(0))} + \delta\, \mathrm{Tail}(f_+; 0, R/2) \qquad \forall \delta \in (0,1]\,,$$

whereas nonnegative supersolutions satifies

$$\mathrm{Tail}(f_+, 0, R) \leq c \sup_{B_{3R/2}(0)} f\,.$$

By a covering argument and choosing $\delta > 0$ small enough it is possible to reabsorb the tail term and to prove the full Harnack inequality after combination with the weak one

$$\fint_{B_R(0)} f(v)\,\mathrm{d}v \leq c \inf_{B_{R/2}(0)} f$$

- Di Castro, Kuusi & Palatucci, *JFA* (2014)
- Cozzi, *JFA* (2017) (via De Giorgi classes).

Similarly happens for parabolic-type problems

- Strömqvist *Ann. IHP-C* (2019) (for **global solutions**)
- Kassmann & Weidner, *Anal. PDE* (2024) + *Duke* (2024)

Counterexample to the strong Harnack inequality

Theorem (Kassmann & Weidner, *Adv. Math.* (2024))

There exist a constant $c_\circ > 0$ *such that for every* $\varepsilon \in (0, \frac{1}{4})$ *there exists a solution* $f_\varepsilon : \mathbb{R}^{2n} \mapsto [0,1]$ *to*

$$v \cdot \nabla_x f_\varepsilon + (-\triangle_v)^s f_\varepsilon = 0 \qquad \text{for } (x,v) \in B_1(0) \times B_1(0)\,,$$

such that for $\xi := (\frac{1}{2}e_n, 0) \in \mathbb{R}^{2n}$, *it holds*

$$f_\varepsilon(\xi) \leq c_\circ \varepsilon^{n(1+2s)-2s} f_\varepsilon(0).$$

In particular, $f_\varepsilon(0)/f_\varepsilon(\xi) \to \infty$ *as* $\varepsilon \to 0$.

- It is an effect purely originating from the combination of the nonlocality of the diffusion combined with the anisotropy of the transport.
- It is very surprising when compared to all the previous literature dealing with local kinetic equations
- No obstruction to regularity
- The classical L^∞-L^2-estimate fails in general. Moreover, the L^∞-L^2 estimate remains false if an L^p-norm of the tail is added on the right-hand side if $p < \frac{n(1+2s)}{2s}$.

Theorem (Anceschi, Palatucci & Pic., preprint (2025))

Let $\Omega := (t_1, t_2) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$ *be a domain and* $s \in (0,1)$. *Assume that* $f \in \mathcal{W}$ *is a weak subsolution to*

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{L}_K f + h \quad \text{in } \Omega.$$

Then, there exists $p^* \equiv p^*(n,s) > 2$ *such that if, for some* $p > p*$, *it holds* $\mathrm{Tail}(f_+; B) \in L^p_{loc}((t_1, t_2) \times \Omega_x)$, *for all* $B \Subset \Omega_v$, *and* $h \in L^p_{\rm loc}(\Omega)$, *then, for any* $Q_r(z_0) \Subset \Omega$ *and any* $\delta \in (0,1]$, *it holds*

$$\sup_{Q_{\frac{r}{2}}(z_0)} f \quad \leq \quad c \left(\frac{\langle v_0 \rangle}{\delta r^{n+3s}} \right)^{\beta} \|f_+\|_{L^2(Q_r(z_0))} + \|h\|_{L^p(Q_r(z_0))} \tag{3}$$

$$+\delta \| \mathrm{Tail}(f_+; B_{r/2}(v_0)) \|_{L^p(U_r(t_0, x_0))},$$

where $\beta \equiv \beta(n,s,p) > 0$ *and* $c \equiv c(n,s,\Lambda,p) > 0$. *Moreover,* $\beta, c \nearrow \infty$ *as* $p \searrow p^*$.

- We want to built a De Giorgi-type argument for proving the supremum estimate.
- The procedure is based on a combination of a Sobolev inequality and an energy estimate.

Remark

Because of the strong degeneracy of the involved equations we can not rely on embedding theorems given by the function space $\mathcal{W}$ itself.

- We rely on the fundamental solutions of the fractional Kolmogorov equation to transfer regularity in all the variables

Proposition

For any $\sigma > 0$, let g be a weak solution of

$$\begin{cases} (\partial_t + v \cdot \nabla_x) g + (-\triangle_v)^s g \leq (-\triangle_v)^{s/2} h_1 + h_2 & \text{in} \quad [-\sigma^{2s}, 0] \times \mathbb{R}^{2n}, \\ g(-\sigma^{2s}, x, v) = g_\text{o}(x, v) & \text{in} \quad \mathbb{R}^{2n}. \end{cases}$$

Assume $h_1, h_2 \in L^2([-\sigma^{2s}, 0] \times \mathbb{R}^{2n})$. Then

$$\|g\|_{L^q([-\sigma^{2s},0]\times\mathbb{R}^{2n})} \lesssim \|g_\text{o}\|_{L^2(\mathbb{R}^{2n})} + \|h_1\|_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})} + \|h_2\|_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})}$$

for any $2 \leq q \leq 2 + \frac{2s}{n(1+s)}$.

Proof of the gain integrability

Assume $z_\circ = 0$. Fix $0 < \varrho < r < 1$, and define $\sigma := \varrho + (r-\varrho)2^{-3}$ and

$$\begin{cases} \psi = \psi(x,v) \in C_c^\infty(B_{((\varrho+\sigma)/2)^{1+2s}} \times B_{(\varrho+\sigma)/2}) \\ \psi \equiv 1 \text{ on } B_{\varrho^{1+2s}} \times B_\varrho \text{ and } 0 \le \psi \le 1\,, \\ |\nabla_v \psi| \le c/(r-\varrho) \text{ and } |(v+v_\circ)\cdot\nabla_x\psi| \le c\langle v_\circ\rangle/(r-\varrho)^{1+2s} \end{cases}$$

Then, the function $g := (f-\kappa)_+\psi$ satisfies

$$\begin{aligned} &(\partial_t + v\cdot\nabla_x)g + (-\triangle_v)^s g \\ &\quad \le (-\triangle_v)^s g + (f-\kappa)_+\big(v\cdot\nabla_x\psi\big) + \psi\chi_{\{f>\kappa\}}h + \mathcal{L}_K g \\ &\qquad + \left(\mathrm{p.v.}\int_{\mathbb{R}^n} \frac{(f-\kappa)_+(w)\big(\psi(w)-\psi(v)\big)}{|v-w|^{n+2s}}\,\mathrm{d}w\right)\chi_{\{f>\kappa\}} \\ &\quad =: (-\triangle_v)^{s/2}h_1 + h_2. \end{aligned}$$

Remark

Observe that the right-hand side involves fractional differentiation with respect to the v-variable only, and these are the directions where we got some regularity estimates from energy estimates.

Proof of the gain integrability

Estimating the right-hand side depending on long and short interactions as well as on the ranges of fractional index $s \in (0,1)$, we get

$$\begin{aligned}
&\|h_1\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})} + \|h_2\|^2_{L^2([-\sigma^{2s},0]\times\mathbb{R}^{2n})} \\
&\quad \leq \frac{c\langle v_\circ\rangle^2}{(r-\varrho)^{2(n+3s)}}\|(f-\kappa)_+\|^2_{L^2(Q_r)} + \frac{c}{(r-\varrho)^2}\int_{U_r}[(f-\kappa)_+]^2_{s,2,B_\sigma}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \frac{c\,|Q_r\cap\{f>\kappa\}|^{1-\frac{2}{p}}}{(r-\varrho)^{2(n+2s)}}\|\mathrm{Tail}((f-\kappa)_+;B_r)\|^{\frac{2}{p}}_{L^p(U_r)} \\
&\qquad + c\,|Q_r\cap\{f>\kappa\}|^{1-\frac{2}{p}}\|h\|^{\frac{2}{p}}_{L^p(Q_r)},
\end{aligned}$$

The proof finishes combining the above ones with classical energy estimates

$$\sup_{t\in[-T,0]}\|g(t,\cdot,\cdot)\|^2_{L^2_{x,v}} + \int_{-T}^0 \|g(t,\cdot,\cdot)\|^2_{L^2_x H^s_v}\,\mathrm{d}t \lesssim \|g\|^2_{L^2} + \langle\text{nonlocal tail terms}\rangle$$

Local gain of integrability

Theorem (Anceschi, Palatucci & Pic., preprint (2025))

Let $\Omega := (t_1, t_2) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$ *be a domain and* $s \in (0,1)$. *Assume that* $f \in \mathcal{W}$ *is a weak subsolution. Then,*

$$f \in L^q_{loc}(\Omega) \quad \forall\, q \in \left[2, 2 + \frac{2s}{n(1+s)}\right].$$

Furthermore, for any $p > 2$ *such that* $\mathrm{Tail}(f_+; B) \in L^p_{loc}((t_1, t_2) \times \Omega_x)$, *for any* $B \Subset \Omega_v$ *and* $h \in L^p_{loc}(\Omega)$, *and any* $Q_r \Subset \Omega$, *the following estimate does hold, for* $0 < \varrho < r$,

$$\begin{aligned}
&\|(f-\kappa)_+\|_{L^q(Q_\varrho)} \\
&\leq \frac{c\,\langle v_0\rangle}{(r-\varrho)^{2(n+3s)}}\|(f-\kappa)_+\|_{L^2(Q_r)} + \frac{c\,|Q_r \cap \{f > \kappa\}|^{\frac{1}{2}-\frac{1}{p}}}{r-\varrho}\|h\|_{L^p(Q_r)} \\
&+ \frac{c\,|Q_r \cap \{f > \kappa\}|^{\frac{1}{2}-\frac{1}{p}}}{(r-\varrho)^{2(n+2s)}}\|\mathrm{Tail}((f-\kappa)_+; B_r)\|_{L^p(U_r)}\,,
\end{aligned}$$

for any $\kappa \in \mathbb{R}$.

De Giorgi iteration and the supremum estimate

We built a nonlinear recursive argument. For $j \in \mathbb{N}$ define

$$R_j := \frac{1}{2}\left(1 + \frac{1}{2^j}\right) R \quad \text{and} \quad \kappa_j := \left(1 - \frac{1}{2^j}\right)\kappa_{\rm o} \quad \text{for } R, \kappa_{\rm o} > 0$$

and

$$Y_j := \kappa_{\rm o}^{-2} \int_{Q_{R_j}} (f - \kappa_j)_+^2 \, \mathrm{d}z$$

Applying the local gain of integrability we thus arrive at

$$Y_{j+2} \lesssim \delta^{-1} \boldsymbol{b}^j Y_j^{1+\alpha} \quad \text{with} \quad \boldsymbol{b} > 1\,,$$

where

$$\begin{cases} \delta \in (0,1] \quad \text{(arbitrary)} \\ \alpha := 1 - \frac{2}{p} - \frac{2}{q} \quad \text{for } p > 2,\ q \in \left(2, 2 + \frac{2s}{n(1+s)}\right]\,, \end{cases}$$

Choosing

$$\begin{cases} p > 2 + \frac{2n(1+s)}{s} \\ \kappa_{\rm o} \approx \left(\frac{\langle v_{\rm o}\rangle}{\delta r^{n+3s}}\right)^{\frac{1}{\alpha}} \|f_+\|_{L^2(Q_r)} + \|h\|_{L^p(Q_r)} + \delta \,\|\,\mathrm{Tail}(f_+; B_{r/2})\|_{L^p(U_r)}\,, \end{cases},$$

we obtain that

$$\alpha > 0 \quad \text{and} \quad Y_{\rm o} \lesssim \boldsymbol{b}^{-\frac{2}{\alpha^2}}$$

so that a classical iteration argument yields that $Y_j \to 0$ as $j \to \infty$, giving the desired result.

Theorem (Anceschi, Palatucci & Pic., preprint (2025))

Let $\Omega := (t_1, t_2) \times \Omega_x \times \Omega_v \subset \mathbb{R}^{1+2n}$ *be a domain,* $Q_2(0) \Subset \Omega$*, and* $s \in (0,1)$*. Assume that* $f \in \mathcal{W}$ *is a globally nonnegative weak solution to*

$$(\partial_t + v \cdot \nabla_x) f = \mathcal{L}_K f + h \quad \text{in } \Omega.$$

Then, there exists $p^* \equiv p^*(n,s) > 2$ *such that if, for some* $p > p*$*, it holds* $\mathrm{Tail}(f_+; B) \in L^p_{loc}((t_1,t_2) \times \Omega_x)$*, for all* $B \Subset \Omega_v$*, then there exists* $R_\mathrm{o} \in (0,1)$ *depending only on* n *and* s *such that*

$$\sup_{Q^-} f \lesssim \inf_{Q^+} f + \| \mathrm{Tail}(f; 0, R_\mathrm{o}/2) \|_{L^p(U_{R_\mathrm{o}}(-1+R_\mathrm{o}^{2s}, 0))},$$

where

$$Q^+ := (-R_\mathrm{o}^{2s}, 0] \times B_{R_\mathrm{o}^{1+2s}} \times B_{R_\mathrm{o}}$$
$$\textit{and} \quad Q^- := (-1, -1 + {R_\mathrm{o}}^{2s}] \times B_{{R_\mathrm{o}}^{1+2s}} \times B_{R_\mathrm{o}}$$

Proof of the strong Harnack inequality

We combine the weak Harnack inequality with the following covering Lemma

Lemma

There exist constants $c_* \equiv c_*(s) \in (0,1)$ *and* $\gamma \equiv \gamma(s) \geq 1$ *such that, for any* $1/2 \leq \varrho < r \leq 1$ *and any* $z_o \in \mathbb{R}^{1+2n}$*, it holds*

$$Q_{c_*(r-\varrho)^\gamma}(z) \subset Q_r(z_o) \qquad \forall z \in Q_\varrho(z_o)\,,$$

For $1/2 \leq \sigma' < \sigma \leq 1$, by the L^∞-L^2 estimate we have, for $\gamma_1, \gamma_2 > 0$ (universal)

$$\sup_{Q_{\sigma' R_o}(-1+R_o^{2s},0,0)} f \leq \frac{c(\delta)\|f\|_{L^\zeta(Q^-_{R_o})}}{[(\sigma-\sigma')R_o]^{\gamma_1}} + \left(c\delta + \frac{2-\zeta}{2}\right) \sup_{Q_{\sigma R_o}(-1+R_o^{2s},0,0)} f$$

$$+ \frac{c(\delta)}{[(\sigma-\sigma')R_o]^{\gamma_2}} \|\mathrm{Tail}(f_+; 0, R_o/2)\|_{L^p(U_{R_o}(-1+R_o^{2s},0))}$$

Choosing $\delta \in (0,1)$ sufficiently small, we reabsorb the supremum on the left-hand side and conclude with the weak Harnack inequality.

- Our Harnack formulation does not contradict the counterexample built via the sequence $\{f_\varepsilon\}$ since

$$\frac{\sup f_\varepsilon}{\inf f_\varepsilon + \| \mathrm{Tail}(f_\varepsilon)\|_{L^p}} < \infty \quad \text{as } \varepsilon \searrow 0 \,.$$

- The lower bound of the exponent p is given by the highest integrability range achievable via convolution with the fundamental solution.

- Weak solutions are not required to have finite p-tail. However, in accordance with the Boltzmann case, boundedness on the mass trivially implies finite p-tail, as, e. g.
  - Silvestre *CMP* (2016): Theorem 1.1 – 1-2
  - Imbert, Mouhot & Silvestre, *JEP* (2020): Formula (1.3)
  - Imbert & Silvestre *JEMS* (2020: Section 1.3
  - Imbert & Silvestre, *JAMS* (2022): Assumption 1.2

THANK YOU